1. There are 12 seats in a circle, with one person in each seat. Simultaneously, every person moves to one of the two seats adjacent to them, with equal probability. If the probability that no two people move to the same seat can be written as $\frac{p}{q}$ for relatively prime integers p and q, find $p + q$.

   Answer: 1025

   Solution: Number the seats $1, 2, 3, 4, \ldots, 12$. Then people on odd-numbered seats move to even-numbered seats and vice versa, so we can look at these people separately. For the people on odd-numbered seats, if the person at 1 moves to 2, then the person at 3 must move to 4, the person at 5 must move to 6, and so on. So once we choose where person 1 goes, there is only 1 way for the other 5 people, so $\frac{1}{32}$ probability. Similarly, there is a $\frac{1}{32}$ probability that none of the people on even-numbered seats go to the same seat, so then the total probability is $\frac{1}{32} \cdot \frac{1}{32} = \frac{1}{1024}$, and the answer is $1 + 1024 = \boxed{1025}$.

2. Find the sum of all positive integers n such that $\lfloor \sqrt[n]{1010} \rfloor < \lfloor \sqrt[n]{2020} \rfloor$.

   Answer: 25

   Solution: This occurs if and only if there is some integer k with $1010 < k^n \leq 2020$. This is true for $n = 1, 2, 3, 4, 5$, as we can take $k = 1011, 32, 11, 6, 4$ respectively. For $n = 6$, we have $3^6 = 729 < 1010$ and $4^6 = 4096 > 2020$, so this does not work. Since $3^7 > 2020$, then for $n = 7, 8$, and 9 we have $2^n < 1010$ and $3^n > 2020$, so these n do not work. For $n = 10$ we can take $k = 2$. For larger n, we have $2^n > 2020$, so they cannot work. So then the sum of all n is $1 + 2 + 3 + 4 + 5 + 10 = \boxed{25}$.

3. Find the number of integers k for which k^{666} has less than 1000 digits.

   Answer: 63

   Solution: A nonnegative integer has less than 1000 digits iff it is less than 10^{999}, which is 1 followed by 999 zeroes. Therefore, we have

$$k^{666} < 10^{999}$$
$$k < 10^{\frac{3}{2}}$$
$$k < \lfloor\sqrt{1000}\rfloor.$$

   We know $\sqrt{1000}$ is strictly between 31 and 32, so $-31 \leq k \leq 31$, which contains $\boxed{63}$ solutions.

4. If $x + y = 6$ and $x^3 + y^3 = 108$, find $x^5 + y^5$.

   Answer: 2376

   Solution: First, by expanding out $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$, we have $216 = 108 + 18xy$, so $xy = 6$. Then, we do the same for the fifth power:

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$\to (x + y)^5 = (x^5 + y^5) + 5(xy)(x^3 + y^3) + 10(xy)^2(x + y)$$

$$\to x^5 + y^5 = (x + y)^5 - 5(xy)(x^3 + y^3) - 10(xy)^2(x + y)$$

$$= 6^5 - 5(6)(108) - 60(6)^2(6)$$

$$= 7776 - 3240 - 2160 = \boxed{2376}$$

5. If x, y, z are real numbers selected uniformly at random in the interval $[0, 2]$, and the probability that $x + y + z \le \sqrt[3]{5}$ can be written as $\frac{p}{q}$ for relatively prime integers p and q, find $p + q$.

Answer: 53

Solution: Choosing x, y, z according to these constraints is equivalent to choosing a random point in the cube defined by $0 \le x, y, z \le 2$. Note that $x + y + z \le \sqrt[3]{5}$ cuts out a trirectangular tetrahedron with vertices $(0, 0, 0), (\sqrt[3]{5}, 0, 0), (0, \sqrt[3]{5}, 0), (0, 0, \sqrt[3]{5})$. Since the volume of a tetrahedron with base area b and height h is $\frac{1}{3}bh$, we have $\frac{1}{3}(\frac{1}{2}(\sqrt[3]{5})^2)(\sqrt[3]{5}) = \frac{5}{6}$. The probability that a point within the cube falls within the tetrahedron is the volume of the tetrahedron divided by the volume of the cube, which is $\frac{5}{48}$. Our answer is $48 + 5 = \boxed{53}$.

6. Let ABC be a triangle with $AB = 6, AC = 8, BC = 10$. Construct rectangles $BCDE$, $ABFG$, and $CAHI$ outside triangle ABC. If $BE = 4$ and points D, E, F, G, H, I lie on a circle, and the area of the circle is $k\pi$, find k.

Answer: 41

Solution: Let O be the center of the circle. Then O is on the perpendicular bisectors of DE, FG, and HI. But these are the perpendicular bisectors of BC, AB, and AC, so then O is the circumcenter of ABC. Since $\angle BAC = 90^\circ$, this means O is the midpoint of BC. So then $OD^2 = 4^2 + 5^2 = 41$, and the area is 41π, so $k = \boxed{41}$.

7. The maximal odd divisor of an integer n is $\frac{n}{2^k}$, where k is the largest possible value such that $\frac{n}{2^k}$ is an integer. Find the sum of the maximal odd divisors for positive integers between 1 and 100, inclusive.

Answer: 3344

Solution: We use

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \tag{1}$$

extensively throughout this problem.

We group numbers by their number of factors of 2:

Factors of 2	Values	Corresponding Maximal Odd Divisors	Sum
0	$1, 3, 5, \ldots 99$	$1, 3, 5, \ldots 99$	50^2
1	$2, 6, 10, \ldots 98$	$1, 3, 5, \ldots 49$	25^2
2	$4, 12, 20, \ldots 100$	$1, 3, 5, \ldots 25$	13^2
3	$8, 24, 40 \ldots 88$	$1, 3, 5, \ldots 11$	6^2
4	$16, 48, 80$	$1, 3, 5$	3^2
5	$32, 96$	$1, 3$	2^2
6	64	1	1^2

Adding, we get

$$50^2 + 25^2 + 13^2 + 6^2 + 3^2 + 3^2 + 1^1 = \boxed{3344}$$

8. Gauss's teacher was unable to stump Gauss with the sum of the first 100 numbers, so he turned to the multiplicative version of it: factorials. He defines $F_n(x)$ to be the largest integer p such that n^p divides x, and asks Gauss to find $F_{10}(100^{100} \cdot 99^{99} \cdot 98^{98} \cdots 3^3 \cdot 2^2 \cdot 1^1) - F_{10}(100! \cdot 99! \cdot 98! \cdots 3! \cdot 2! \cdot 1!)$. What does he get?

Answer: 176

Solution: Using Legendre's Formula for valuations,

$$F_{10}(100! \cdot 99! \cdot 98! \cdots 3! \cdot 2! \cdot 1!) = (20+4) + (19+3) + (19+3) + \cdots (0+0) + (0+0) + (0+0).$$

Regrouping, we get

$$(20 + 19 \cdot 5 + 18 \cdot 5 + 17 \cdot 5 + \cdots 2 \cdot 5 + 1 \cdot 5 + 0 \cdot 5) + (4 + 3 \cdot 25 + 2 \cdot 25 + 1 \cdot 25 + 0 \cdot 24)$$

$$= \left(20 + 5 \cdot \frac{19 \cdot 20}{2} + 4 + 75 + 50 + 25\right) = 20 + 950 + 154 = 1124.$$

Next, we find $F_{10}\left(100^{100} \cdot 99^{99} \cdot 98^{98} \cdots 3^3 \cdot 2^2 \cdot 1^1\right)$. We essentially need to find $F_5(100^{100} \cdot 99^{99} \cdot 98^{98} \cdots 3^3 \cdot 2^2 \cdot 1^1)$, as the powers of 5 will be the limiting factor (not the factors of 2). However, all exponents $x : x \not\equiv 0 \pmod 5$ do not provide any factors of 5. Thus, we really want to find $F_5(100^{100} \cdot 95^{95} \cdot 90^{90} \cdots 15^{15} \cdot 10^{10} \cdot 5^5$. We can edit this to make it such that the bases of the exponents only have powers of 5 (essentially, 100 would be replaced with 25 and 15 would be replaced with 5). The result is $F_5\left(25^{100} \cdot 5^{95} \cdot 5^{90} \cdots 5^{15} \cdot 5^{10} \cdot 5^5\right)$. This yields

$$F_5(5^{5 \cdot \frac{20 \cdot 21}{2}} \cdot 5^{250}) = F_5(5^{1300}) = 1300.$$

Thus, our answer is $1300 - 1124 = \boxed{176}$.

9. Ashwin picks a point $P = (a, b)$ in the interior of the unit circle uniformly at random and draws the line l_1 connecting P and $(0, 1)$, and the line l_2 connecting P and $(0, -1)$. The probability that either the slope of line l_1 or the slope of l_2 is less than 3 in absolute value can be written as $1 - \frac{p}{q\pi}$, where p and q are relatively prime positive integers. Find $p^2 + q^2$.

Answer: 13

Solution: For each of $(0, 1)$ and $(0, -1)$, we draw the two lines containing that point of slope 3 and -3. The line through $(0, 1)$ with slope -3, as well as the line through $(0, -1)$ with slope 3, intersects the x-axis at $\left(\frac{1}{3}, 0\right)$; and the other two lines intersect the x-axis at $\left(-\frac{1}{3}, 0\right)$. Notice that any point outside of the rhombus formed by these four lines will satisfy the property in the problem. The rhombus has area $\frac{2}{3}$, so for a randomly selected point $P(a, b)$, the probability that either the slope of line l_1 or the slope of l_2 is less than 3 in absolute value is

$$\frac{\pi - \frac{2}{3}}{\pi} = 1 - \frac{2}{3\pi}$$

This gives us an answer of $4 + 9 = \boxed{13}$.

10. The decimal notation of $\frac{1}{17}$ is the repeating decimal

$$0.\overline{058823529A1176BC},$$

which has 16 digits in the repetend, or the part which repeats. Find the three-digit integer $\overline{ABC}$.

Answer: 447

Solution: Since there are 16 digits in the repetend, the repetend multiplied by 17 must equal 0.999...99 with 16 9's.

This means that 058823529A1176BC multiplied by 17 must equal 9999999999999999. Looking at the units digits, $C \cdot 17$ ends in 9. Therefore, C must be 7. Looking at the last 2 digits, $B7 \cdot 17$ ends in 99. This can be written as the equation $17(10B+7) \equiv 99 \pmod{100}$. Solving this gives $B = 4$.

Since 058823529A1176BC is 9999999999999999 divided by 17, it must be a multiple of 9. The sum of its digits must be a multiple of 9. Substituting $B = 4$ and $C = 7$ gives the solution $A = 4$.

The final answer is $\boxed{447}$.

11. Sarah is bored one day and decides to walk around the unit circle. She starts at $P(1,0)$ and walks a distance of π in the counterclockwise direction. She then turns around, and walks a distance of $\frac{2\pi}{3}$. She turns around again, and walks a distance of $\frac{2\pi}{9}$. She turns around again, and walks a distance of $\frac{4\pi}{27}$. After the nth turn, she walks a distance of $\frac{2^{\frac{n+1}{2}}}{3^n}\pi$ if n is odd, and a distance of $(\frac{2}{9})^{n/2}\pi$ if n is even. After walking for a long time, Sarah becomes arbitrarily close to a point Q, where $m\angle POQ = \theta$. If $\angle POQ = \frac{p}{q}\pi$ for relatively prime integers p and q, find $p+q$.

Answer: 10

Solution: Finding θ boils down to evaluating two geometric series. We have

$$\theta = \pi - \frac{2\pi}{3} + \frac{2\pi}{9} - \frac{4\pi}{27} + \frac{4\pi}{81} - \frac{8\pi}{243} + \cdots = \left(\pi + \frac{2\pi}{9} + \frac{4\pi}{81} + \cdots\right) - \left(\frac{2\pi}{3} + \frac{4\pi}{27} + \frac{8\pi}{243} + \cdots\right)$$

$$= \frac{\pi}{1-\frac{2}{9}} - \frac{\frac{2\pi}{3}}{1-\frac{2}{9}} = \frac{\frac{\pi}{3}}{\frac{7}{9}} = \frac{3\pi}{7}.$$

Thus, our answer is $3 + 7 = \boxed{10}$.

12. Find the integer k between 0 and 99 inclusive such that $k \equiv \frac{2^{2020}-1}{15} \pmod{100}$.

Answer: 5

Solution: Notice that $15 = 2^4 - 1$. The expression can then be written as a geometric series

$$2^{2016} + 2^{2012} + 2^{2008} + \cdots + 2^4 + 1$$

Since we are only looking for the last 2 digits, and the last 2 digits of powers of 2 cycle with period 20, we can calculate the last 2 digits of $2^4, 2^8, 2^{12}, 2^{16}$, and 2^{20} (which are 16, 56, 96, 36, and 76), and substitute them into the expression. Notice that $2^0 \equiv 1 \not\equiv 76 \pmod{100}$.

$$36 + 96 + 56 + 16 + 76 + 36 + \cdots + 16 + 1$$

Notice that $36 + 96 + 56 + 16 + 76 \equiv 80 \pmod{100}$. Since these 5 numbers repeat 100 times from 2^{2016} to 2^{20}, they can be disregarded since they add to $8000 \equiv 0 \pmod{100}$. Now all that's left is to calculate the last 2 digits of $2^{16} + 2^{12} + 2^8 + 2^4 + 1$.

$$36 + 96 + 56 + 16 + 1 = \boxed{5} \pmod{100}.$$

13. Let $f(n)$ be the number of ordered triples of nonnegative integers (a, b, c) with $2a + b + c = n$. If

$$\sum_{n=0}^{\infty} \frac{f(n)}{2^n}$$

can be written as $\frac{p}{q}$ for relatively prime integers p and q, find $p + q$.

Answer: 19

Solution: Instead of summing over n, we sum over a, b, c. Then each triple (a, b, c) contributes $\frac{1}{2^{2a+b+c}}$ to the sum, so then the sum is equal to

$$\begin{aligned}\sum_{a,b,c\geq 0} \frac{1}{2^{2a+b+c}} &= \left(\sum_{a\geq 0} \frac{1}{2^{2a}}\right) \cdot \left(\sum_{b\geq 0} \frac{1}{2^{b}}\right) \cdot \left(\sum_{c\geq 0} \frac{1}{2^{c}}\right) \\ &= \frac{4}{3} \cdot 2 \cdot 2 = \frac{16}{3},\end{aligned}$$

and the answer is $16 + 3 = \boxed{19}$.

14. Let ABC be an acute triangle with orthocenter H and circumcenter O. Let M be the midpoint of arc BC not containing A. Suppose that $AB = 10, AC = 12$, and $HA = HM$. Find the square of the length of BC.

Answer: 124

Solution: Note that $OA = OM$, so both AHM and AOM are isosceles. Additionally, $AH \parallel OM$ because both are perpendicular to BC, so then $\angle HAM = \angle AMO$. This means HAM and OAM are congruent isosceles triangles, so $AH = OM = R$, where R is the circumradius of ABC. If we let E be the foot of the perpendicular from B to AC, we can calculate

$$AH = \frac{AE}{\cos{(90 - C)}} = \frac{AE}{\sin C} = \frac{c \cos A}{\sin C} = 2R \cos A.$$

Since $AH = R$, this means $\cos A = \frac{1}{2}$. Then by the Law of Cosines,

$$BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos A = 100 + 144 - 2 \cdot 10 \cdot 12 \cdot \frac{1}{2} = \boxed{124}.$$

15. Find the number of functions $f : \{1, 2, \ldots, 15\} \to \{1, 2, \ldots, 15\}$ satisfying the property that $\gcd(f(x), f(y)) = \gcd(x, y)$ for all $x \neq y$.

Answer: 117

Solution: First, note that if $x \leq 7$, then for all $kx \leq 15$ with $k > 1$, we have

$$\gcd(f(x), f(kx)) = \gcd(x, kx) = x.$$

So then x must divide $f(kx)$ for all $k \geq 1$ with $kx \leq 15$.

This also means that if $x \leq 7$ does not divide y, then x cannot divide $f(y)$: this is because $x \mid f(x)$, so then if $x \mid f(y)$ we would have

$$x \mid \gcd(f(x), f(y)) = \gcd(x, y),$$

which is impossible.

So then if $x \leq 7$, we have $x \mid f(y)$ if and only if $x \mid y$.

Now for all y divisible by any number $2 \leq x \leq 7$, we cannot have primes $p > 7$ dividing $f(y)$, as then $f(y) \geq px \geq 2 \cdot 8$. So for all such y, the set of primes dividing $f(y)$ is the same as the set of primes dividing y, which means the only choices we get to make are the exponents of these primes, and additionally, $4 \mid f(y)$ if and only if $4 \mid y$. This means $f(x) = x$ for $x = 2, 5, 6, 7, 10, 12, 14-$ we only have a choice when we are deciding whether to have 3 or 9, and 4 or 8, and this only is a choice for $3, 4, 8, 9$ because otherwise the number would become too large.

Then we can have $(f(4), f(8))$ be $(4, 8), (8, 4)$, or $(4, 4)$ - they must both be 4 and 8, and cannot both be 8. This is 3 choices - no other $f(x)$ can be divisible by 8, so this does not affect the other choices. Similarly, we have 3 choices for $f(3)$ and $f(9)$ $((3, 9), (9, 3), (3, 3))$.

Finally, with $f(1), f(11), f(13)$, these cannot be divisible by any primes $p \leq 7$, so they must all be $1, 11$, or 13. Additionally, we cannot have 11 or 13 repeated. Any such choice works, since none of the other $f(x)$ are divisible by 11 or 13. There is 1 choice using three 1's, 6 choices using two 1's, and 6 choices using one 1, so 13 choices total.

So in total, there are $3 \cdot 3 \cdot 13 = \boxed{117}$ such functions.