Time limit: 10 minutes per set
Instructions: For each set, you will individually solve two short answer questions.
No calculators.

1. Find the positive integer x such that

$$\log_5 12, \log_5 x, \log_5 108$$

is an arithmetic sequence in that order.

Answer: 36

Solution: If real numbers a, b, and c are an arithmetic sequence in that order, then they satisfy $b - a = c - b$, or $b = \frac{a+c}{2}$. Therefore, $\log_5 x = \frac{\log_5 12 + \log_5 108}{2} = \log_5 \sqrt{12 \times 108} \implies x = \sqrt{12 \times 108} = \boxed{36}$.

2. A rhombus has side length 10 and area 80. A circle is inscribed in the rhombus. If the area of the circle can be written as $k\pi$, find k.

Answer: 16

Solution: Consider the rhombus drawn in Figure 1. Since the total area of the rhombus is 80, the area

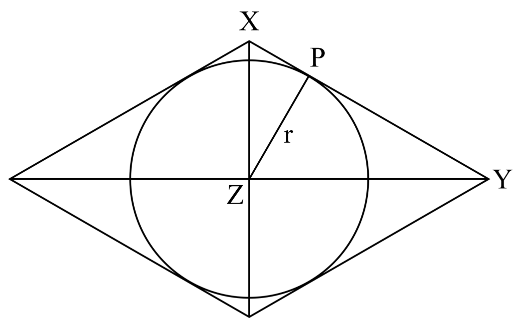

Figure 1: Rhombus with side length 10 and area 80

of triangle XYZ is a quarter of 80, or 20. Now, we know that radii to points of tangency are perpendicular to tangent lines, so $ZP \perp XY$. We calculate the area of triangle XYZ:

$$\begin{aligned}[XYZ] &= \frac{1}{2}(XY)(ZP) \\ &= \frac{1}{2}(10)(r) \\ &= 5r.\end{aligned}$$

Therefore, $5r = 20$, so $r = 4$. The area of the circle is $\pi r^2 = 16\pi$, and the answer is $\boxed{16}$.

3. If $f(x)$ is a linear function and $f(f(f(x))) = 8x + 693$, find $f(1)$.

Answer: 101

Solution: Let $f(x) = ax + b$. Then,

$$f(f(x)) = a(ax + b) + b = a^2x + ab + b,$$

and

$$f(f(f(x))) = a(a^2x + ab + b) = a^3x + a^2b + ab + b = 8x + 693.$$

Equating the x coefficient yields $a = 2$. Now, equating the constant terms, we have $a^2b + ab + b = 693$. Plugging in $a = 2$, we get $7b = 693 \to b = 99$. Our original function is $f(x) = 2x + 99$, and $f(1) = \boxed{101}$.

4. Find the number of permutations of $1, 2, 3, \dots, 10$ such that for all integers x with $1 \le x \le 5$, in the permutation x appears to the left of $2x$.

   Answer: 37800

   Solution: Consider the sets $\{1, 2, 4, 8\}, \{3, 6\}, \{5, 10\}, \{7\}, \{9\}$. Within each of these sets, the numbers must occur in increasing order, and any such permutation works. So then we can choose which slots belong to which set. There are $\binom{10}{4}$ ways to choose where $1, 2, 4, 8$ go, then $\binom{6}{2}$ ways to choose $3, 6$, then $\binom{4}{2}$ ways to choose $5, 10$, and then 2 ways to choose 7. So the number of ways is

$$\binom{10}{4} \cdot \binom{6}{2} \cdot \binom{4}{2} \cdot 2 = 210 \cdot 15 \cdot 6 \cdot 2 = \boxed{37800}.$$

5. Moor takes a triangular sheet of paper and lays it over another (not necessary congruent) triangular sheet of paper. The resulting polygon formed by the two triangles together has n sides. What is the sum of all possible values of n?

   Answer: 64

   Solution: We can use some simple casework, working downward from a hexagram, or working upward from a triangle entirely encased in the other triangle. The answer is $3+4+5+6+7+8+9+10+12 = \boxed{64}$.

6. How many values x satisfy $0 \le x < 1001$ and $x^3 - 1000x^2 - 9x + 992 \equiv 0 \pmod{1001}$?

   Answer: 27

   Solution: Notice that $-1000 \equiv 1 \pmod{1001}$ and $992 \equiv -9 \pmod{1001}$, so we can write the equation as

$$x^3 + x^2 - 9x - 9 = (x + 1)(x^2 - 9) = (x + 1)(x + 3)(x - 3) \equiv 0 \pmod{1001}.$$

   Since $1001 = 7 \cdot 11 \cdot 13$, we have $x \equiv 3, 4, 6 \pmod 7$, $x \equiv 3, 8, 10 \pmod{11}$, and $x \equiv 3, 10, 12 \pmod{13}$. By the Chinese Remainder Theorem, each combination of remainders modulo 7, 11, and 13 gives a unique value modulo 1001, so we have $3 \cdot 3 \cdot 3 = \boxed{27}$ solutions.

7. Maira is interested in constructing quadratics out of each other and starts with a relatively simple example. She takes two quadratics: $(x + m)^2$ and $(x + m + 2)^2$, and takes their average (adding both together and dividing the sum by 2). She takes her new quadratic and finds the roots r_m and s_m. Finally, she calculates the remainder of $\sum_{m=1}^{2020}(r_m^2 + s_m^2)$ when it is divided by 1000. What value does she get?

   Answer: 580

Solution: The average of the two quadratics is $x^2+(2m+2)x+(m^2+2m+2)$. Using Vieta's formulas, we know that

$$r_m^2+s_m^2=(r_m+s_m)^2-2r_ms_m=(2m+2)^2-m^2+2m+2=2m^2+4m.$$

So, Maira is now calculating

$$\sum_{m-1}^{2020}(2m^2+4m).$$

Using the formulas for the sum of squares $1+4+9+\cdots+n^2=\frac{n(n+1)(n+2)}{6}$ and natural numbers $1+2+3+\cdots+n=\frac{n(n+1)}{2}$, we get

$$2\cdot\frac{2020\cdot 2021\cdot 4041}{6}+4\cdot\frac{2020\cdot 2021}{2}\equiv 20\cdot 21\cdot 413+2\cdot 20\cdot 21 \pmod{1000}$$

$$\equiv 20\cdot 7\cdot 41+20\cdot 42\equiv 20(287+42)\equiv 20(329) \pmod{1000}$$

$$\equiv 3290\cdot 2\equiv 290\cdot 2\equiv \boxed{580} \pmod{1000}.$$

8. For positive integers n and b, let $f_b(n)$ denote the sum of the digits of the base b representation of n. How many positive integers $n\leq 2020$ satisfy $f_2(n)=f_4(n)$?

Answer: 63

Solution: If we have the base 4 representation of n, we convert to base 2 by writing each digit as a 2-digit base 2 number, so we would write $00, 01, 10, 11$ in place of $0, 1, 2, 3$. The sum of the digits of this base 2 number is always less than or equal to the sum of the digits of the base 4 number, with equality if and only if the digit is 0 or 1. So then the n which work are the ones whose digits in base 4 are all 0 or 1. Note that $4^5<2020<2^{11}$, so we need n to have at most 6 digits in base 4, and we can add leading zeroes so that n has exactly 6 digits. Then we have 2 choices for each of these digits, so 64 choices total. This includes 0, which is not allowed. Other than this, the maximum number we get is $1+4+4^2+\ldots+4^5=\frac{4^6-1}{3}<2020$, so all such numbers work. So then there are $\boxed{63}$ values of n.

9. Two elements are selected at random from the set $\{1,2,3,\cdots,2020\}$. If the expected value of the larger of the two elements is $\frac{m}{n}$ in lowest terms, find $m+n$.

Answer: 4045

Solution: There's a total of $\binom{2020}{2}$ possible pairs. There is one way for the larger element to be 2, 2 ways for the larger element to be 3, and so on, up to 2019 ways for the larger element to be 2020. Thus, the sum of the larger elements over all possible pairs is

$$1\cdot 2+2\cdot 3+\cdots+2019\cdot 2020$$

Using the identities $\sum_{i=1}^n i=\frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^2=\frac{n(n+1)(2n+1)}{6}$, we have

$$1\cdot 2+2\cdot 3+\cdots+2019\cdot 2020=\frac{2019\cdot 2020\cdot 2021}{3}$$

The expected value is the sum of the larger elements of the pairs, divided by the number of possible pairs. So we have:

$$\frac{\frac{2019\cdot 2020\cdot 2021}{3}}{\frac{2020\cdot 2019}{2}} = \frac{4042}{3}$$

Thus, our answer is $4042 + 3 = \boxed{4045}$

10. Sam writes all positive divisors of 5040 (including itself) on a whiteboard. Then, in one turn, he picks one of the numbers on the whiteboard uniformly at random, and then erases that number and all multiples of it. He keeps doing this until all numbers are erased. If the expected number of turns taken by this process is $\frac{m}{n}$ in lowest terms, find $m+n$.

Answer: 1667

Solution: The number of turns is the number of numbers which are chosen. By Linearity of Expectation, if each $k \mid 5040$ has probability P_k of being chosen, then the expected number of chosen numbers is $\sum_{k|5040} P_k$.

To find P_k, consider all $d(k)$ divisors of k, including itself. Then k gets erased the first time one of these divisors gets chosen, and none of the other divisors can get erased until one of these is chosen. This means all numbers on this list are equally likely to be the first one chosen, so then the probability that k is chosen first is $\frac{1}{d(k)}$.

So then we want to find $\sum_{k|5040} \frac{1}{d(k)}$. We can factor $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, so since $d(k)$ is multiplicative, we can write this as

$$\left(\frac{1}{d(1)} + \frac{1}{d(2)} + \frac{1}{d(2^2)} + \frac{1}{d(2^3)} + \frac{1}{d(2^4)}\right) \cdot \left(\frac{1}{d(1)} + \frac{1}{d(3)} + \frac{1}{d(3^2)}\right) \cdot \left(\frac{1}{d(1)} + \frac{1}{d(5)}\right) \cdot \left(\frac{1}{d(1)} + \frac{1}{d(7)}\right)$$
$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) \cdot \left(1 + \frac{1}{2} + \frac{1}{3}\right) \cdot \left(1 + \frac{1}{2}\right)^2$$
$$= \frac{1507}{160},$$

and the expected number of turns is $\frac{1507}{160}$, yielding an answer of $\boxed{1667}$.